CAUCHY'S FURTHER CONTRIBUTIONS TO MATHEMATICS: THE EXISTENCE THEOREMS IN ORDINARY DIFFERENTIAL EQUATIONS

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Abstract
Cauchy was a mathematician who gave pure mathematics a great and important impetus; for fecundity and variety of production, he can be compared only to Euler: his writings, published during forty-seven years of continuous work, in separate volumes or in scientific collections, are about 789. To prevent this immense work from being lost, the Académie des Sciences in Paris began publishing the "Oeuvres complétes" as early as 1882, which is not yet finished. In this paper, we will examine one interesting innovations Cauchy made to mathematics of all time, the existence theorems in ordinary differential equations.

Keywords: differential equations, theorems

1. The existence theorems in ordinary differential equations

Ordinary differential equations originated in the eighteenth century as a direct response to physical problems. On the other hand, dealing with particular strings in physics that are more complicated, in the course of research on vibrating elements, the mathematicians arrived at partial differential equations. In the nineteenth century the roles of these two theories were in a certain way exchanged. Efforts to solve partial differential equations with the method of separation of variables led in fact to having to solve ordinary differential equations. Furthermore, since partial differential equations were expressed in various coordinate systems, the resulting ordinary differential equations were strange and not soluble in closed form. Mathematicians were thus reduced to looking for solutions developed in infinite series which are now known as special functions or higher transcendent functions, in contrast to elementary transcendent functions such as$\sin x e^x \log x$. After much research on a huge variety of ordinary differential equations, some in-depth theoretical studies were then devoted to the various types of these equations. These theoretical researches also differentiate nineteenth-century studies from those of the eighteenth. The contributions of the new century were so vast that, as in the case of partial differential equations, we cannot hope to review all the most important developments. The fact that existence problems had been neglected for so long is partly due to the fact that differential equations arose in physical or geometric contexts in which it was intuitively clear that they admitted solutions. Cauchy[1] was the first to consider the problem of the existence of solutions of differential equations and to give two methods to solve it. The first method that applies to the equation:

[1] Augustin-Louis Cauchy (1789 - 1857) was a French mathematician and engineer. He started the project of the formulation and rigorous proof of the theorems of infinitesimal analysis based on the use of the notions of limit and continuity. He also made important contributions to the theory of complex variable functions and to the theory of differential equations. The systematic nature and level of these works place him among the fathers of mathematical analysis.

$$y' = f(x, y)$$

it was created between 1820 and 1830 and is summarized in the Exercises d'analyse. This method, whose existence can already be found in Euler, uses the same idea underlying the concept of integral as the limit of a sum. Cauchy wants to state that there is one and only one function $y = f(x)$that you satisfy $y' = f(x, y)$ and the initial condition$y_0 = f(x_0)$. To do this, Cauchy divides the interval $(x_0, x)n$in parts $\Delta x_0, \Delta x_1, \Delta x_2, \Delta x_3 \ldots\ldots \Delta x_{n-1}$ and pones:

$$y_{i+1} = y_i + f(x_i, y_i)\Delta x_i$$

where x_i is any value of$x$ contained in Δx_ithen by definition we will have:

$$y_n = y_0 + \sum_{i=0}^{n-1} f(x_i\ , y_i)\,\Delta x_i$$

and Cauchy shows that, for n tending to infinity, yn converges to a single function:

$$y = y_0 + \int_{x_0}^{x} f(x_i\,, y_i)\,\Delta x_i$$

which satisfies the equation $y\,' = f\,(x,y)$and the given initial condition. Cauchy hypothesized that $f\,(x, y)$ and $f(y)$were continuous for all the real values of $x$and $y$contained in the rectangle determined by the intervals $(x_0, x)(y_0\,, y)$and. In 1876 Rudolph Lipschitz[2] undermined the hypotheses of the theorem. In 1876 Rudolph Lipschitz undermined the hypotheses of the theorem. Its essential condition was that all points (x, y_1) and (x, y_2) contained in the rectangle:

$$|x - x_0| \leq |y - y_0| \leq b$$

that is, for any two points with the same abscissa, a constant exists k such that:

$$|\,f\,(x, y_1) \;-\; f(x, y_2)\,| \;<\; k\,(y_1 \;-\; y_2).$$

This condition is known as the Lipschitz condition and the existence theorem is called the Cauchy-Lipschitz theorem. The second Cauchy method to establish the existence of solutions of differential equations, the method of dominant or major functions, has wider applications than the first and was applied by Cauchy himself in the complex domain. The method was presented in a series of works published in the "Comptes Rendus" of the years 1839-1842. The method was called by Cauchy "calcul des limites" because it provides lower limits within which it is certain that the solution whose existence is established converges. The method was simplified by Briot and Bouquet and their version became standard. To illustrate the method, let's apply it to the equation:

$$y' \;=\; f\;(x, y)$$

where$f$è analitico in $xy$and. The existence theorem which must be stated as follows:

[2] Rudolph Otto Sigismund Lipschitz (1832-1903) was a German mathematician. He devoted his studies to number theory, Bessel functions, the Fourier series, the calculus of variations and differential equations. He also dealt with hydrodynamics and analytical mechanics.

(1) $$\frac{dy}{dx} = f(x,y)$$

the function $f\,(x,y)$ is analytic in the neighborhood of the point$P_0 = (x_0, y_0)$, then the differential equation has a unique solution $y(x)$which is analytic in a neighborhood of $x_0$and which reduces to y_0 when$x = x_0$. The solution can be represented by the series:

(2) $$y = y_{0+}y'(x - x_0) + \frac{y_0''}{2!}(x - x_0)^2 + \frac{y_0'''}{3!}(x - x_0)^3 + \dots\dots$$

where $y_0' \frac{dy}{dx}$is calculated in (x, y_0) and similarly for$y_o'', y_0''', \dots$ and where the derivatives are determined by subsequent derivation of the original differential equation in which $y$is considered as a function of$x$. The demonstration method uses first of all the fact that, since$f(x,y)$ is analytic in the neighborhood of (x_0, y_0) for convenience we will assume it coincides with the origin$(0{,}0)$, there is a circle with a radius "a" and center $x_0 = 0$and a circle of radius "b" and center $y_0 = 0$ where$f(x,y)$ is analytic. Therefore $f(x,y)$has an upper bound$M$ for all the values of $xy$and of the respective circles. Now, the very method with which the series (2) was obtained guarantees that it formally satisfies the equation:

(3) $$\frac{dx}{dy} = f(x,y)$$

The problem is to prove that the series converges. For this purpose, the majorizing function is considered:

(4) $$F(x,y) = \sum \frac{M}{a^p b^q} x^p y^q$$

which is the development of the function:

(5) $$F(x,y) = \frac{M}{\left(1-\frac{x}{a}\right)\left(1-\frac{y}{b}\right)}$$

It is then shown that the solution of the equation:

(6) $$\frac{dY}{dx} = f(x,Y)$$

date from the series:

$$Y = Y_0' + Y_0'' \frac{x^2}{2!} + Y_0'' \frac{x^3}{3!} + \cdots$$

The latter was obtained from (6) in the same way that (2) was obtained from (1). So if (7) converges, the same happens for (2). To prove that (7) converges, solve (6) explicitly using the value of F given by (5) and prove that the series expansion of the solution, which must coincide with (7), is convergent. The method does not by itself determine the precise radius of convergence of the series that gives the y. Numerous efforts were therefore devoted to finding a

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demonstration that the radius can be extended. None of these, however, gives the entire domain of convergence and are therefore of little practical importance. A third method for establishing the existence of solutions of ordinary differential equations, probably known to Cauchy, was first published by J. Liouville[3] in Journal de Mathematique in 1838 in relation to a second order equation. This is the method of successive approximations which is today attributed to Emile Picard[4], he published his studies in the Journal de Mathematique in 1893, because he gave it a general form.

For the equation:

$$y' = f(x, y)$$

where $f\,(x, y)$is analytic in realxy and whose solution $y = y\,(x)$ must pass through the point(x_0, y_0) the method consists in introducing the sequence of functions:

$$y_1(x) = y_0 + \int_{x_0}^{x} f(t, y_0)\ dt$$

$$y_1(x) = y_0 + \int_{x_0}^{x} [t, y_1(t)]\ dt$$

....

....

$$y_n(x) = y_0 + \int_{x_0}^{x} [t, y_{n-1}(t)]\ dt$$

and in proving that $y_n\,(x)$ tends to a limit $y(x)$ which is the only continuous function of x which satisfies the differential equation and the condition:

$y(x_0) = y_0$

The method as it is usually presented today only assumes that $f(x,y)$satisfies the Lipschitz condition. The method was extended by Picard in his 1893 work to second order equations and

[3] Joseph Liouville (1809 - 1882) was a French mathematician. Son of a soldier who survived the campaigns of Napoleon Bonaparte and settled in Toul in 1814, he graduated from the *École Polytechnique*. Subsequently he began the *École nationale des ponts et chaussées*, without however obtaining the diploma for health reasons and for having developed an inclination to an academic career rather than an engineering one. After a few years as an assistant in various institutes, in 1838 he was appointed professor at the École polytechnique. In addition to his mathematical talent, Liouville displayed remarkable organizational skills. He founded the *Journal de Mathématiques Pures et Appliquées* , still highly reputed today. He was the first to read the unpublished works of Évariste Galois, to publish them in his newspaper in 1846, and to recognize their importance. Liouville was also politically engaged for a short time: a member of the constituent assembly in 1848, he left it after an electoral defeat in 1849.

[4] Charles Émile Picard (1856 - 1941) was a French mathematician and academic. Despite the death of his father, director of a silk factory, during the siege of Paris in 1870, he was able to study at the Napoleon Lyceum (the current Henry IV Lyceum). Coming second in the competition to enter the École polytechnique and first in the one for the École norma supérieure, he opted for the latter and enrolled in it in 1877. Assistant first in Paris and then in Toulouse, he became a research doctor in 1881. In the same year, he married Marie, daughter of his colleague Charles Hermite. Three of their five children died during the First World War . Émile Picard quickly made a name for himself in the world of mathematicians, proving a difficult theorem based on the singularity of holomorphic functions, later completed by Gaston Julia. This job earned him a first candidacy to become a member of the Académie des sciences, an election postponed to 1889 due to his young age. In 1885 he became professor at the science faculty of the University of Paris, occupying the chair of differential calculus in place of Joseph-Alfred Serret, then he taught mathematical analysis and algebra. He was also a lecturer at the École centrale des Arts et Manufactures, from 1894 to 1937, teaching Mechanics to more than ten thousand future engineers.

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has also been extended to the case where x and y are complex. The various methods described above were applied not only to ordinary differential equations but also to systems of differential equations with complex variables. Thus, Cauchy extended his second type of existence theorem to systems of first-order ordinary differential equations i "n" independent variables. He also extended this method of "calcul des limites" to systems in the complex domain. Cauchy's result says the following, given the system of equations:

$$\frac{dy_k}{dx} = f_k(x, y_0, \dots \dots, y_{n-1}) \qquad k = 0,1,2, \dots \dots, n-1$$

Where $f_0, \dots \dots \dots, f_{n-1}$ they are monogenous functions of their arguments, such that it is possible to develop them around the initial values:

$$x = \xi \qquad y_0 = \eta_0, \dots \dots \qquad y_{n-1} = \eta_{n-1}$$

in series of positive integer powers of:

$$x - \xi \qquad y_0 - \eta_0, \dots \dots \qquad y_{n-1} - \eta_{n-1}$$

then they exist n power series of x-ξ converging around of x=ξ,

which you substitute in place of $y_0, \dots \dots \dots \dots \dots . y_{n-1}$ in the following equation:

$$dy_n = \frac{f_x}{d_x}(x, y_0, \dots \dots \dots . y_{n-1}) \qquad k = 0,1,2, \dots n-1$$

2. Conclusion

These power series are unique, provide a smooth solution of the system and assume the given initial values. In this general form the theorem can be found in Cauchy's memoir entitled " Mémoire sur l'emploi du nouveau calcul, appelé calcul des limites, dans l'intégration d'un système d'equations différentielles". The idea was therefore to be content with establishing the existence of a solution and obtaining it around a point of the complex plane. Weierstrass[5] achieved the same result in the same year (1842), but did not publish it until his Werkes came out in 1894. The interest in nonlinear equations has strengthened in the twentieth century, the applications have extended from astronomy to the problems of communications, servomechanics, control systems and electronics systems. The study today has moved from the qualitative level to the quantitative surveys.

[5] Weierstrass Karl Theodor Wilhelm (1815 –1897) was a German mathematician often cited as the "father of modern analysis". Despite leaving university without a degree, he studied mathematics and trained as a school teacher, eventually teaching mathematics, physics, botany and gymnastics. He later received an honorary doctorate and became professor of mathematics in Berlin. Among many other contributions, Weierstrass formalized the definition of the continuity of a function, proved the intermediate value theorem and the Bolzano–Weierstrass theorem, and used the latter to study the properties of continuous functions on closed bounded intervals.

Bibliography

Belhoste B.: Augustin-Louis Cauchy. A Biography New York, 1991.

Belhoste B.: Cauchy. Un mathématicien légitimiste au XIXe siècle Paris, 1985.

Bliss Gilbert A.: *The reduction of singularities of plane curves by birational transformation*, "Bulletin of the American mathematical society", 29, 1923, pp. 161-168.

Bogolyubov A. N.: Augustin Cauchy and his contribution to mechanics and physics (Russian), Studies in the history of physics and mechanics, 'Nauka' Moscow, 1988, 179-201.

Bottazzini U.: *Il calcolo sublime. Storia dell'analisi matematica da Euler a Weierstrass*, Torino, Gray Jeremy J.: *Algebraic geometry between Noether and Noether. A forgotten chapter in the history of algebraic geometry*, "Revue d'histoire des mathématiques", 3, 1997, pp. 1-48.

Cutland N.- Kessler C.:, E Kopp and D Ross, On Cauchy's notion of infinitesimal, British J. Philos. Sci. 39 (3) 1988, 375-378.

Dahan-Dalmedico A.: L'intégration des équations aux dérivées partielles linéaires à coefficients constants dans les travaux de Cauchy 1821-1830, Études sur Cauchy (1789-1857), Rev. Histoire Sci. 45 (1), 1992, 83-114.

Dahan-Dalmedico A.: La mathématisation de la théorie de l'élasticité par A L Cauchy et les débats dans la physique mathématique française (1800-1840), Sciences et Techniques en Perspective 9 ,1984-85, 1-100.

Dahan-Dalmedico A.: Les travaux de Cauchy sur les substitutions : étude de son approche du concept de groupe, Arch. Hist. Exact Sci. 23 (4), 1980/81, 279-319.

Dubbey J. M.: Cauchy's contribution to the establishment of the calculus, Ann. of Sci. 22 1966, 61-67.

Fisher G.: Cauchy's variables and orders of the infinitely small, British J. Philos. Sci. 30 (3), 1979, 261-265.

Fisher G.: Cauchy and the infinitely small, Historia Mathematica 5, 1978, 313-331.

Freudenthal, H. Did Cauchy plagiarise Bolzano, Archive for History of Exact Sciences 7, 1971, 375-392.

Gario P.: Cauchy's theorem on the rigidity of convex polyhedra (Italian), Archimede 33 (1-2),1981, 53-69.

Gilain C.: Cauchy et le cours d'analyse de l'École Polytechnique, Bull. Soc. Amis Bibl. École Polytech. (5) ,1989, 1-145.

Grabiner J. V.:, The origins of Cauchy's theory of the derivative, Historia Math. 5 (4), 1978, 379-409.

Grabiner J. V.:, Who gave you the epsilon? Cauchy and the origins of rigorous calculus, Amer. Math. Monthly 90 (3) ,1983, 185-194.

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Grattan-Guinnes I.: Bolzano, Cauchy and the 'new analysis' of the early nineteenth century, Archive for History of Exact Sciences 6, 1970, 372-400.

Grattan-Guinness I.: On the publication of the last volume of the works of Augustin Cauchy, Janus 62 (1-3), 1975, 179-191.

Grattan-Guinness I.: The Cauchy - Stokes - Seidel story on uniform convergence again: was there a fourth man?, Bull. Soc. Math. Belg. Sér. A 38 ,1986, 225-235.

Gray J.: , Cauchy-elliptic and abelian integrals, Études sur Cauchy (1789-1857), Rev. Histoire Sci. 45 (1) ,1992, 69-81.

Gui Z.L.- Zhao D. F., Cauchy: an outstanding mathematician (Chinese), J. Central China Normal Univ. Natur. Sci. 23 (4), 1989, 597-60

Hawkins Thomas W.: *New light on Frobenius' creation of the theory of group characters*, "Archive for history of exact sciences", 12, 1974, pp. 217-243.

Koetsier T.: Cauchy's rigorous calculus: a revolution in Kuhn's sense?, Nieuw Arch. Wisk, 1987, 335-354

Ostrowski A. M.: *Über den ersten und vierten Gaussschen Beweis des Fundamentalsatzes der Algebra*, Berlin, Springer, 1933.

Purkert W.: *Ein Manuskript Dedekinds über Galois-theorie*, "NTM. Schriftenreihe für Geschichte der Naturwissenschaften, Technik und Medizin", 13, 1976, pp. 1-16. Rider R. E.: *A bibliography of early modern algebra, 1500-1800*, Berkeley, Office for History of Science and Technology, University of California, 1982.

Rider R. E., *A bibliography of early modern algebra, 1500-1800*, Berkeley, Office for History of Science and Technology, University of California, 1982.