International Journal of Advanced Engineering and Management Research

Vol. 6, No. 05; 2021

ISSN: 2456-3676

CAUCHY: HIS CONTRIBUTION TO THE STUDY OF DETERMINANTS AND NUMBER THEORY

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Abstract

In this article, we will examine two interesting contributions that Cauchy has made to mathematics of all time: determinants and number theory. In linear algebra, the determinant of a square matrix is a number that describes some algebraic and geometric properties of the matrix. The determinant is a powerful tool used in various fields of mathematics: first of all in the study of systems of linear equations, then in multidimensional calculus (for example in the Jacobian), in tensor calculus, in differential geometry, in combinatorial theory, etc. The theory of numbers is also of fundamental importance. This topics addressed by the great mathematician are very important for mathematics of all times. They were of fundamental importance for subsequent studies.

Keywords: determinants, number theory, history of mathematics

1. Determinants

Determinants owe their origin to studies on the solution of systems of linear equations. This problem, together with the theory of elimination, the theory of coordinate transformation, the change of variables in multiple integrals, the solution of the systems of differential equations that arise in the study of planetary motion, the reduction to normal form of quadratic forms in three or more variables and bundles of forms(a bundle is the set of all shapes $A + \varphi B$, where A e B are given forms and φ is a parameter) gave rise to various new uses of determinants. These nineteenth-century researches derive directly from those of Cramer[1], Bézout[2], Vandermonde[3],

[1] Gabriel Cramer (1704 - 1752), professor of philosophy and mathematics in Geneva. Cramer demonstrated great skills in mathematics from a young age. At the age of 18 he received his doctorate and at 20 he became co-lecturer in mathematics at the University of Geneva, sharing the chair with the almost contemporary Jean-Louis Calandrini. In 1728 he proposed a solution to the St. Petersburg Paradox and came very close to the concept of the Theory of Expected Utility, theorized 10 years later by Daniel Bernoulli. He was a pupil of Johann Bernoulli. He dealt with studies on algebraic curves and their singularities, as well as on determinants. In this regard, in 1750 he wrote the treatise on analytical geometry entitled *Introduction a l'analyse des courbes algebriques*. He is responsible for the rule of solving a system of n equations in n unknowns. The proof of Cramer's rule for linear systems with two and three unknowns is due to the Scotsman Colin Maclaurin.

[2] Étienne Bézout (1730 - 1783) was a French mathematician. Becoming a mathematician after reading Euler's works, Bézout taught in military schools, also becoming an examiner in competitions for admission to the Navy; he was assigned the task of writing a textbook for these courses, which, under the title of Cours de mathématiques à l'usage des Gardes du Pavillon et de la Marine, was published in four volumes between 1764 and 1769, and later, after becoming a successor to Charles Étienne Louis Camus as an extension of the Corps of Artillery, such as Cours complet de mathématiques à l'usage de la marine et de dell'arteillerie. This book was very popular, so much so that it was imitated and translated into several languages: it was used for example at Harvard University. In 1769 he became associated with the French Academy of Sciences. Bézout deals with algebra, in particular in the field of equations: in 1764 he published Sur le degré des équations résultantes de l'évanouissement des inconnues, a paper on the solution of systems of linear equations, defining by recurrence a quantity equivalent to the determinant of the coefficients of the system. In his work Théorie générale des équations algébraiques, of 1779, he proved Bézout's theorem, which states that two algebraic curves of degree (respectively) m and n intersect in general in m · n points. He also formulated the Bézout identity for polynomials.

[3] Vandermonde Alexandre-Théophile (1735 - 1796) French mathematician. He also had strong interests and skills in music and chemistry, and collaborated with E. Bézout and A. Lavoisier. He devoted himself to mathematics starting in 1770 and, the following year, he was admitted to

Lagrange[4] and Laplace[5]. The word "determinant", used by Gauss[6] to indicate the discriminant of the quadratic form:

$$ax^2 + 2bxy + cy^2$$

it was applied by Cauchy[7] to the determinants that had already appeared in the eighteenth century. The arrangement of the elements in the form of a square and the two-index notation are also due to Cauchy.

Thus, a third order determinant is written in the form (vertical bars were introduced by Cayley[8] in 1841):

$$\begin{vmatrix} a11 & a12 & a13 \\ a21 & a22 & a23 \\ a31 & a32 & a33 \end{vmatrix}$$

In this work Cauchy gave the first systematic and almost modern treatment of determinants. One of the main results contained therein is the multiplication theorem for determinants. Lagrange had already given this theorem in the case of third-order determinants, but, since the lines of his determinant were formed by the coordinates of the vertices of a tetrahedron, he was not

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the Académie des sciences in Paris. His paper Mémoire sur la résolution des équations (Memory on the resolution of equations, 1771) concerns symmetric functions and the resolution of cyclotomic polynomials. Another work of his entitled Mémoire sur des irrationnelles de différens ordres avec une application au cercle (Memory on irrationals of different order with an application to the circle, 1772) concerns combinatorial analysis, while the writing Mémoire sur l'élimination (Memory on Elimination, 1772) deals with the foundations of the theory of determinants. His name is associated with the particular determinant that bears his name.

[4] Lagrange Joseph-Louis (1736-1813), was an Italian mathematician and astronomer active, in his scientific maturity, for twenty-one years in Berlin and twenty-six in Paris. He is unanimously considered among the greatest and most influential European mathematicians of the eighteenth century; his innovative contributions to mathematical physics are also noteworthy. His most important work is the Mécanique analytique, published in 1788, with which rational and analytical mechanics are conventionally born. In mathematics, he is remembered for his contributions to number theory, for having been among the founders of the calculus of variations (deducing it, in his "Mecánique analytique", through that theoretical formulation of rational mechanics known as Lagrangian mechanics), for the results in the field of differential equations and infinitesimal analysis, as well as for having been one of the pioneers of group theory and classical field theory. In the field of celestial mechanics, he conducted research on the phenomenon of lunar libration and, later, on the movements of the satellites of the planet Jupiter; he investigated, with the rigor of mathematical calculation, the problem of the three bodies and their dynamic equilibrium; he also devoted himself to studies of natural sciences. His pupils were Jean-Baptiste Joseph Fourier, Giovanni Plana and Siméon-Denis Poisson.

[5] Laplace Pierre-Simon (1749 - 1827), was a French mathematician, physicist, astronomer and nobleman. He was one of the leading scientists of the Napoleonic period, in 1799 appointed minister of the interior by Napoleon, who in 1806 conferred on him the title of count of the Empire, then also appointed marquis in 1817, after the restoration of the Bourbons. He made fundamental contributions in various fields of mathematics, physics, astronomy and probability theory and was one of the most influential scientists of his time, also for his contribution to the affirmation of determinism. He put the final spin on mathematical astronomy by summarizing and extending the work of his predecessors in his five-volume work Mécanique Céleste (Celestial Mechanics) (1799-1825). This masterpiece transformed the geometric study of mechanics, developed by Newton, into one based on mathematical analysis.

[6] Gauss Johann Friedrich Carl (1777-1855) was a German mathematician, astronomer and physicist, who made decisive contributions in mathematical analysis, number theory, statistics, numerical calculus, differential geometry, geodesy, geophysics, magnetism, electrostatics, astronomy and optics. Sometimes referred to as "the Prince of mathematicians" (Princeps mathematicorum) as Euler or "the greatest mathematician of modernity" (as opposed to Archimedes, considered by Gauss himself as the greatest mathematician of antiquity), he is counted among the most important mathematicians in history having contributed decisively to the evolution of the mathematical, physical and natural sciences. He defined mathematics as "the queen of the sciences".

[7] Cauchy Augustin Louis (1789-1857) was a French mathematician and engineer. He initiated the project of the formulation and rigorous demonstration of the theorems of infinitesimal analysis based on the use of the notions of limit and continuity. He also made important contributions to the theory of complex variable functions and to the theory of differential equations. The systematic nature and level of these works place him among the fathers of mathematical analysis.

[8] Cayley Arthur (1821 -1895) was an English mathematician who made a strong contribution to the growth of pure mathematics in the Anglo-Saxon world.

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stimulated to generalize it. In Cauchy the general theorem, expressed in modern notation, states that:

$$|a_{ij}| * |b_{ij}| = |c_{ij}|$$

where $|a_{ij}|$ and $|b_{ij}|$ they are determinants of order n and $c_{ij} = \sum a_{ik} b_{kj}$

In other words, the term at the intersection of the i-th row and the j-th column of the product is the sum of the products of the corresponding elements of the i-th row of $|a_{ij}|$ and the j-th column of $|b_{ij}|$. This theorem had been stated and not satisfactorily proved by. Binet[9] in 1812. Cauchy not only proved the theorem, but he also gave a better statement. Cauchy to explain the theorem began with the n elements or numbers $a_1, a_2, a_3 \dots\dots\dots a_n$ and formed the product of these elements with the differences between distinct elements:

$$a_1, a_2, a_3, \dots a_n (a_2 - a_1)(a_3 - a_1) \dots (a_n - a_1)(a_3 - a_2) \dots (a_n - a_2) \dots (a_n - a_{n-1})$$

He then defined the determinant as the expression obtained by transforming each power index into a second lower index; he then arranged the n² different quantities that appeared in this determinant in a non-framework dissimilar to the matrices used today:

$$a_1 * 1, a_1 * 2, \dots a_1 * n$$
$$a_2 * 1, a_2 * 2, \dots a_2 * n$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$a_n * 1, a_n * 2, \dots a_n * n$$

Thus arranged, the n quantities of its determinant formed what was called "a symmetric system of order n". He then defined as conjugate terms those elements whose lower indices were in reverse order, and he called principal terms those terms which were self-conjugated; finally, the product of the terms of what we now call the principal diagonal was called the principal product by him. Later in his memoir Cauchy gave other rules for determining the sign of a development term, using circular substitutions. The eighty-four pages that made up Cauchy's memoir of 1812 do not represent all that the French mathematician wrote on the subject of determinants. He later found many opportunities to use determinants in various situations. In an 1815 memoir on wave

[9] Jacques Philippe Marie Binet (1786-1856) was a French mathematician and astronomer. Binet graduated from the École polytechnique as a student in 1804; as soon as he graduated, in 1806, he worked for the Ponts et Chaussées department but the following year he returned to the École polytechnique as a repeater of descriptive geometry. Later he was professor of mechanics, then inspector of studies. In 1823 he succeeded Jean-Baptiste Delambre in the chair of Astronomy at the Collège de France. Like Cauchy, with whom he was a friend, Binet was a staunch Catholic and a supporter of the Bourbon family's pretender to the French throne. The government in July removed him from his duties at the École polytechnique, but he retained his posts at the Collège de France. His works on pure mathematics, mechanics and astronomy are published in the École polytechnique journal and in the Journal of Liouville. To him we owe important works on Euler's phi function, on the study of expressions that I respect from the law of large numbers, on the fundamental properties of homofocal second degree surfaces, which he first discovered, on the movements of the planets, on the finite difference equations linear for which he formulated an interesting theory. His works on matrix calculus led him to the expression of the nth term of the Fibonacci sequence. In the field of astronomy, his kinematic formulas give the expression in polar coordinates of the velocity and acceleration of bodies subject to a central acceleration, such as the planets of the solar system.

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propagation he applied the concepts and terminology of determinants to a geometric problem and also to a physical problem. Cauchy stated the following theorem in this regard: if A, B, C are the lengths of three edges of a parallelepiped, and if the projections of these lengths on the x, y, z axes of an orthogonal coordinate system are:

$$\begin{matrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{matrix}$$

then the volume of the parallelepiped will be given by the following formula:

$$A_1B_2C_3 - A_1B_3C_2 + A_2B_3C_1 - A_2B_1C_3 + A_3B_1C_2 - A_3B_2C_1 = S(\pm A_1B_2C_3)$$

in the same memory, in relation to the propagation of waves, he applied the notation of his theory of determinants to partial derivatives, substituting a condition that required two lines for its expression with the simple abbreviation:

$$S = \left(\pm \frac{dx}{da}\ \frac{dy}{db}\ \frac{dz}{dc}\right) = 1$$

The expression on the left of the equality sign is obviously what is now called "Jacobian" of $x, y, z,$with respect toa, b, c.

Jacobi's name is tied to functional determinants of this form not because he was the first to use them, but because, as a good algorithm builder, he was particularly enthusiastic about the possibilities inherent in determinant notation. It was, in fact, only in 1829 that Jacobi first used the determinants that carry his ninth. We can say that the determinants are only innovations of language, abbreviated expressions of ideas that already existed in a more extensive form. By themselves they do not directly say anything that has not already been said, albeit in a more verbose way, by equations or transformations. However, determinants have not profoundly influenced the course of the history of mathematics, despite their usefulness as compact expressions. However, it must certainly be reiterated that the concept of determinant has proved to be an extremely useful tool and is now part of the mathematics apparatus. We can certainly affirm that the determinants, however, owe their origin to studies on the solution of linear equation systems.

2. The theory of numbers

Cauchy devoted himself to the study of the properties of numbers since his youth, obtaining a result that placed him at the level of Lagrange and Euler[10]; in fact these had shown the decomposition of any whole number into the sum of four squares.

[10] Euler Leonhard, known in Italy as Euler (1707-1783) was a Swiss mathematician and physicist. He is considered the most important mathematician of the eighteenth century, and one of the greatest in history. It is known to be among the most prolific of all time and has made

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Cauchy established the truth of the analogous proposition enunciated by Fermat[11] for any polygonal numbers, in the memoir entitled *Demonstration gènèrale du thèorème de Fermat sur les nombres polygones* read in 1815 in front of the Parisian Académie in which he stated what had already been said by Fermat that every integer is the sum of several "k" "k-gonal" numbers, the general k-gonal number is precisely:

$$n+\frac{(n^2-n)}{\frac{(k-2)}{2}}$$

At a more mature age he still dealt with questions of higher arithmetic, developing, applying and completing concepts, methods and results due to himself and to Gauss; these studies are collected in the extended Mèmoires sur la Thèorie den Nombres that fills the volume XVII dei Mèm. De l'Inst. De France.

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historically crucial contributions in various areas: infinitesimal analysis, special functions, rational mechanics, celestial mechanics, number theory, graph theory. Pierre Simon Laplace seems to have said "Read Euler; he is the teacher of us all". Euler was undoubtedly the greatest supplier of "mathematical denominations", offering his name to an impressive amount of formulas, theorems, methods, criteria, relations, equations. In geometry: the circle, the line and Euler's points relative to the triangles, plus the Euler-Slim relation, which concerned the circle circumscribed to a triangle; in number theory: Euler's criterion and Fermat-Euler's theorem, Euler's indicator, Euler's identity, Euler's conjecture; in mechanics: Euler angles, Euler critical load (due to instability); in the analysis: the Euler-Mascheroni constant, the Euler gamma function; in logic: the Euler-Venn diagram; in graph theory: (again) the Euler relation; in algebra: Euler's method (relative to the solution of the fourth degree equations), Euler's theorem; in differential calculus: Euler's method (concerning differential equations).

[11] Fermat Pierre (1601-1665) was a French mathematician and magistrate. He was among the leading mathematicians of the first half of the seventeenth century and made important contributions to the development of modern mathematics.

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