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Cauchy's Further Contributions to Mathematics: The Divergent Series and Cauchy's Wrong Theorem

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Abstract
In this article we will first highlight Cauchy's contribution to divergent series: a contribution of fundamental importance for mathematics scholars of all time and then stop at the concept as an incomplete result of Cauchy with respect to the hypotheses relating to the continuity of the sum of a series of functions. Although he was not the only seller, this erroneous result remained famous. It is referred to as Cauchy's incomplete theorem.

Keywords: mathematics, divergent series, Cauchy, wrong theorems

1. The divergent series

The fact that since the late nineteenth century a topic such as divergent series was taken seriously indicates how radically mathematicians had revised their conception of the nature of mathematics. While in the first part of the century they accepted the ban on diverging series on the pretext that mathematics should be confined by some internal requirement or by the dictates of nature to a fixed class of correct concepts, towards the end of the century they recognized freedom. to consider whatever ideas seemed to offer any use. We can recall that divergent series were used throughout the eighteenth century, with greater or lesser awareness of their divergence, because they gave a useful approximation of functions when only a few terms were used. After the birth, with Cauchy of rigorous mathematics, most mathematicians neither followed the dictates and rejected the divergent series, not considering them valid. However, some mathematicians continued to defend divergent series, being impressed by their usefulness, both in the calculation of functions and as analytic equivalents of the functions from which they were derived. Cauchy, but also Abel, realized not without concern that by banning divergent series, they were discarding something useful. Cauchy not only continued to use it but also wrote an article entitled *"Sur l'emploi légitime des séries divergentes*" in which, speaking of the Stirling series, he noted that the series, although divergent for all values of x, can be used to calculate the corresponding logarithm when x is large and positive. In fact, he showed that having fixed the number n of the terms considered, the absolute error committed by stopping the sum at the n-th term is less than the absolute value of the next term, and the error becomes smaller as *"x increases".* Cauchy tried to understand why the approximation provided by the series was so good, but without success. The usefulness of divergent series finally convinced mathematicians that there must be some of their characteristics which, considered carefully, would have detected why they provide good approximations. The willingness of mathematicians to continue working with divergent series was undoubtedly reinforced by another influence that had gradually

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penetrated the mathematical atmosphere: non-Euclidean geometry and the new algebras. Mathematicians slowly began to understand that mathematics is the work of man and that Cauchy's definition could no longer be considered a superior necessity imposed by some superhuman power. Divergent series theory has two main themes. The first is the one already briefly described, namely that some of those series can, for a fixed number of terms, approximate a function better and better as the variable increases. The second theme of the theory of divergent series is the so-called summability concept. It is possible to define the sum of a series in completely new ways that associate finite sums to the divergent Cauchy series. In his most important work on the propagation of waves Cauchy said the principle of the stationary phase. This principle states that the most relevant contributions to the integral come from neighborhoods of the stationary points of $h(t)$, that is, of the points where $h'(t) = 0$. This principle applies to evaluate this integral:

$$f(x) = \int_a^b g(t)e^{ixh(t)}dt$$

when t and x is large, then $\left|e^{ixh(t)}\right|$ is constant and Laplace[1]'s method does not apply, in fact Laplace's method is applied when we want to calculate (i).Intuitively this principle is reasonable, because the integrand can be thought of as an oscillating current or wave of amplitude equal to$|g(t)|$. If t is time, the speed of the wave is proportional to $xh'(t)$ and, if $h'(t)$ is different from zero, the speed of the oscillations grows indefinitely as x tends to infinity. The oscillations are then so rapid that for an entire period $g(t)$ is approximately constant and $xh(t)$ is approximately linear, so that the integral extended to a period vanishes. This reasoning falls within the values of t for which $h'(t) = 0$. Therefore the points which probably make the main contributions to the asymptotic value of $f(x)$are the stationary points of $h(t)$. If τ is a value of t for which $h'(\tau) = 0$ and $h''(\tau) > 0$ then we will have:

$$f \to \sqrt{\frac{2\pi}{xh''(\tau)}}\, g(\tau)e^{ixh(\tau)+i^{\pi}/4}$$

to tend to$x$ to infinity. In the early nineteenth century Cauchy and Poisson developed many integrals containing a power series parameter of the parameter. Cauchy used these integrals for

[1]Laplace Pierre-Simon (1749 - 1827), was a French mathematician, physicist, astronomer and nobleman. He was one of the leading scientists of the Napoleonic period, in 1799 appointed minister of the interior by Napoleon, who in 1806 conferred on him the title of count of the Empire, then also appointed marquis in 1817, after the restoration of the Bourbons. He made fundamental contributions in various fields of mathematics, physics, astronomy and probability theory and was one of the most influential scientists of his time, also for his contribution to the affirmation of determinism. He put the final spin on mathematical astronomy by summarizing and extending the work of his predecessors in his five-volume work Mécanique Céleste (Celestial Mechanics) (1799-1825). This masterpiece transformed the geometric study of mechanics, developed by Newton, into one based on mathematical analysis.

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the study of water waves, in optics and in astronomy. Cauchy working on the diffraction of light (he published this study of his in 1842) gave expressions in divergent series for Fresnel[2] integrals

$$\int_0^m \cos\left[\frac{\pi}{2}z^2\right] dz = \frac{I}{2} - N\cos\frac{\pi}{2}m^2 + M\,sin\frac{\pi}{2}m^2$$

$$\int_0^m \sin\left[\frac{\pi}{2}z^2\right] dz = \frac{I}{2} - M\cos\frac{\pi}{2}m^2 + N\,sin\frac{\pi}{2}m^2$$

Where:

$$M = \frac{I}{m\pi} - \frac{I*3}{m^5\pi^3} + \frac{I*3*5*7}{m^9\pi^5} - \ \ldots.\ldots\ldots$$

$$N = \frac{I}{m^3\pi^2} - \frac{I*3*5}{m^7\pi^4} + \ \ldots.\ldots\ldots$$

In the 19th century, other methods were invented for evaluating integrals. Cauchy gave the definition of sum by reiterating that it is the sum obtained by adding more and more terms in the ordinary sense of the word. The construction and acceptance of the theory of divergent series is another striking example of the way in which mathematics has developed. It shows first of all that when a concept or technique proves useful despite its confusing or non-existent logic, stubborn research will reveal a rationale, which is truly an afterthought. It also demonstrates how far mathematicians have gone in recognizing that mathematics is the work of man. The definitions of summability do not correspond to the natural idea of adding more and more terms, which was made rigorous by Cauchy: they are artificial; they serve mathematical purposes, including the mathematical solution of physical problems; and these are now sufficient reasons for admitting them into the domain of legitimate mathematics.

2. An incomplete result of Cauchy

We have described an incomplete result of Cauchy[3] regarding the hypotheses concerning the continuity of the sum of a series of functions. While he was not the only salesman, this erroneous result remained famous. It is referred to as Cauchy's incomplete theorem. Cauchy's original statement was certain that the sum of a series of continuous functions was also a continuous function, but this statement was later proved not true. Something similar had occurred a few

[2]Fresnel Augustin-Jean (1788 -1827) was a French engineer and physicist who carried out important research in the optical field, in particular in physical optics.

[3] Cauchy Augustin Louis (1789 - 1857) was a French mathematician and engineer. He initiated the project of the formulation and rigorous demonstration of the theorems of infinitesimal analysis based on the use of the notions of limit and continuity. He also made important contributions to the theory of complex variable functions and to the theory of differential equations. The systematic nature and level of these works place him among the fathers of mathematical analysis.

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years earlier: Ampere[4] had deluded himself that all continuous functions were differentiable, but subsequent investigations had shown that this was not true. Before describing the foregoing, it should be noted that eighteenth-century mathematicians used the series with individually questionable logical rigor and unclear results towards the end of the century spurred research on series operations. The first mathematicians to use series correctly were Fourier[5], Gauss[6]and Bolzano[7]. Fourier gave a satisfactory definition of a convergent series. Gauss studied the hyper geometric series without addressing the general principles of convergence. Bolzano in 1817 in one of his works already demonstrated that he had the correct definition of the convergence of a succession and a series. The correct definition of the limit of a sequence was given by Wallis[8] in 1685. Cauchy, in the Course d'analyse algebrique treated the convergence of series in a rigorous and broad way. In particular, he stated the convergence criterion relating to successions:

$$(a_n) \text{ is convergent } \quad (\forall\, a_n \varepsilon\, R\, \forall\, n \varepsilon N) \Leftrightarrow \forall\, \varepsilon > 0$$

$$\exists\, \nu(\varepsilon) > 0 \mid \forall\, n, m > \nu(\varepsilon) \Rightarrow |a_n - a_m| < \varepsilon$$

and demonstrated, with regard to the series, the necessary and sufficient condition for their convergence:

$$\sum_{k=0}^{\infty} a_k \text{ is convergent } \Leftrightarrow$$

$$\forall \varepsilon > 0\, \exists\, \nu(\varepsilon) | \quad for \quad n > \nu(\varepsilon) \text{ and } \quad \forall\, p\, \varepsilon\, N \Rightarrow \left|a_{n+1} + \;\ldots\ldots\ldots, a_{n+p}\right| < \varepsilon$$

He also stated the necessary condition for the convergence of a series. Self:

$$\sum_{k=0}^{\infty} a_k \text{ is convergent } \Leftrightarrow \lim_{k \Rightarrow +\infty} a^k = 0$$

and for those with positive terms the criterion of comparison: if it turns out that

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[4]Ampère André-Marie (1775 -1836) was a French physicist. He made fundamental contributions in the study of electrodynamics, developing the first mathematical models for the description of the phenomena of electromagnetism. In his honor, the homonymous unit of measurement for electric current is adopted in the International System. His name appears among the 72 names engraved on the Eiffel Tower.

[5]Fourier Jean Baptiste Joseph (1768 - 1830) was a French mathematician and physicist, best known for his famous series and transform and for his law on the conduction of heat.

[6]Gauss Johann Friedrich Carl (1777 - 1855) was a German mathematician, astronomer and physicist, who made decisive contributions in mathematical analysis, number theory, statistics, numerical calculus, differential geometry, geodesy, geophysics, magnetism, electrostatics, astronomy and optics. Sometimes referred to as "the Prince of mathematicians" (Princeps mathematicorum) as Euler or "the greatest mathematician of modernity" , he is counted among the most important mathematicians in history having contributed decisively to the evolution of the mathematical, physical and natural sciences. He defined mathematics as "the queen of the sciences".

[7]Bolzano Bernard Placidus Johann Nepomuk (1781 - 1848) was a Bohemian mathematician, philosopher, theologian, presbyter and logician who wrote in German, making significant contributions to both mathematics and the theory of knowledge.

[8]John Wallis (1616 - 1703) was an English presbyter and mathematician. Wallis contributed to the development of infinitesimal calculus. Between 1643 and 1689 he was chief cryptographer of the Parliament of the United Kingdom and later of the royal court. He is also credited with introducing the symbol denoting the mathematical concept of infinity.

$$a_k < b_k \Rightarrow \sum_{k=0}^{\infty} a_k \text{ is convergent if it is } \sum_{k=0}^{\infty} b_k$$

and also:

$$\sum_{k=0}^{\infty} b_k \text{ is divergent if it is} \sum_{k=0}^{\infty} a_k$$

and the convergence criterion of the nth root:

if $\lim_{k \Rightarrow +\infty} a^{\frac{1}{k}} < 1 \Rightarrow \sum_{k=0}^{\infty} a_k \text{ it is convergent}$

if instead:

$$\lim_{k \Rightarrow +\infty} a^{\frac{1}{k}} > 1 \Rightarrow \sum_{k=0}^{\infty} a_k \text{ it is divergent}$$

Cauchy also proved that the sum of two convergent series is convergent to the sum of the respective series and the analogous result for the product. Lagrange[9] was the first to specify the remainder of Taylor[10]'s formula, but it was Cauchy, in 1823 and 1829, who observed that the Taylor series converges to the function from which it is derived if the nth remainder tends to zero as n tends to Infinity. Cauchy also gave a different formula for the rest of Taylor's formula. In the Course d’analyse he stated, erroneously, that the sum of a convergent series of continuous functions is continuous; in the Résumé des lecons I affirm that if it works$u_n(x)$constituents of the series are continuous and if the series converges it is possible to integrate term by term, not realizing, however, that further hypotheses were necessary: the uniform convergence of the

[9]Lagrange Joseph-Louis (1736 - 1813), was an Italian mathematician and astronomer active, in his scientific maturity, for twenty-one years in Berlin and twenty-six in Paris. He is unanimously considered among the greatest and most influential European mathematicians of the eighteenth century; his innovative contributions to mathematical physics are also noteworthy. His most important work is the Mécanique analytique, published in 1788, with which rational and analytical mechanics are conventionally born. In mathematics, he is remembered for his contributions to the theory of numbers, for having been among the founders of the calculus of variations (deducing it, in his "Mécanique analytique", through that theory of rational mechanics known as Lagrangian mechanics), for the results in the field of differential equations and infinitesimal analysis, as well as being one of the pioneers of group theory and classical field theory. In the field of celestial mechanics, he conducted research on the phenomenon of lunar libration and, in, on the movements of the satellites of the planet Jupiter; he investigated, with the rigor of mathematical calculation, the problem of the three bodies and their dynamic equilibrium; he also devoted himself to studies of natural sciences. His pupils were Jean-Baptiste Joseph Fourier, Giovanni Plana and Siméon-Denis Poisson.

[10]Taylor Brook (1685 - 1731) English mathematician. A supporter of Newtonian mechanics, he is known for his contributions to the development of differential calculus. He studied law at St. John's College in Cambridge, earning his degree (1709) and doctorate (1714) there, but he never seems to have practiced as a lawyer. Also in 1714 he published his solution to the problem of the center of oscillation of a body, which he had already found in 1708; this delay in publication gave rise to a dispute over priority with Johann Bernoulli. After the Essay on linear perspective (Linear perspective, 1715), Taylor began the publication of his main work, Methodus incrementorum directa et inversa (1715-17), which contains the famous formula on the power series development of functions, now known as Taylor's theorem, and the solution of the vibrating string problem. The importance of the Taylor series was later recognized by J. Lagrange, who placed it at the foundation of differential calculus. In 1712 he was elected a member of the Royal Society of London and from 1714 to 1718 he held the role of secretary.

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series. Abel[11] in 1826 highlighted Cauchy's error relating to the continuity of the sum of a convergent series of continuous functions. In this regard he considered the series:

$$\sin -\frac{1}{2}\sin 2(x) + \frac{1}{3}\sin(3x) - \frac{1}{4}\sin(4x) + \ \dots\dots$$

which is Fourier series expansion of $y = \left(\frac{1}{2}\right)x$ in the interval $[-\pi;\ \pi]$ and therefore represents the periodic function which results to be equal to $y = \left(\frac{1}{2}\right)x$ in each amplitude interval 2π. The sum of the series is discontinuous for $x = (2n+1)\pi,\ n\varepsilon Z$, in fact it converges to $\left(\frac{1}{2}\right)\pi$ for x tending to $(2n+1)\pi$ from left and to $\left(-\frac{1}{2}\right)\pi$ for x tending to $(2n+1)\pi$ from the right. Abel correctly proved Cauchy's erroneous theorem relating to the continuity of the sum of a series of continuous functions using the concept of uniform convergence, but he did not give the right emphasis to the additional hypotheses, that is, to those ideas that would have led Weierstrass[12] to the definition of uniform convergence. In fact Weierstrass arrived at this definitive synthesis but the first who stressed its importance was Heine[13] in a work on trigonometric series.

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[11]Abel, Niels Henrik (1802 - 1829), Norwegian mathematician; his studies are responsible for determining results for the birth of modern mathematics, especially in the field of algebra and the theory of functions. After completing his studies at the University of Christiania (today's Oslo), he spent two years in Paris and Berlin and in 1828, returning to his homeland, he had a position as a teacher. In algebra, his name is famous for the theorem (independently obtained also by the Italian mathematician Paolo Ruffini) which proves the impossibility of solving, by means of elementary algebraic procedures, algebraic equations of degree higher than the fourth. He was responsible for the definition of an important class of transcendent functions, now known as "abelian", generated by the inversion of a particular type of integral (elliptic integral), and of various elements of the analysis (abelian integral) and of the 'algebra (abelian group, abelian manifolds). Abel also gave a systematic guise to the theory of power series and, in particular, clarified the binomial theorem, already formulated by Isaac Newton and Euler, including the cases of irrational and imaginary powers.

[12] Weierstrass Karl (1815 - 1897) was a German mathematician , often called the "father of modern analysis ".

was the first of four children of Wilhem Weierstrass, a government official, and Theodora Vonderforst, who died when he was 12. His interest in mathematics began when he was still a *gymnasium* student . Having converted his father to Catholicism, Weierstrass grew up in Catholic circles, also teaching in various Catholic middle schools. He enrolled at the University of Bonn where, according to his father's wishes, he had to be prepared to hold a government post. Since his studies had to cover the fields of law , economics and finance, he immediately found himself in conflict with his own aspiration to the study of mathematics, attracted by the *Fundamenta Nova* of Karl Gustav Jacob Jacobi . He resolved this conflict by paying little attention to the planned course of study and continuing to study mathematics self-taught and reading Crelle's Journal . The result was to leave the University of Bonn without a degree. After studying mathematics at the University of Münster , a place already very famous at that time for mathematics, he obtained the professorship in Münster . During this period of study, Weierstrass followed the lectures of Christoph Gudermann and developed a keen interest in elliptic functions and abelian functions . The conflict with his father led him to drink and have mental problems; after 1850 Weierstrass suffered for a long period of illness, but was able to publish articles that brought him fame and distinction, so much so that in 1854 he was appointed honorary doctor at the University of Königsberg . In 1857 he obtained the chair of mathematics at the University of Berlin . Although universally honored, a true phobia of publication developed in him, requiring his lectures to circulate among his disciples only in handwritten copies. He had as students Georg Cantor , Felix Klein , Sophus Lie ,Hermann Minkowski . He also gave private lessons to Sofia Kovalevskaya , since women could not enroll in the university. He dealt with rigorously defining the foundations of the analysis, first giving the example of a function that is continuous everywhere but not derivable . Its name is linked to the Weierstrass theorem , the Bolzano-Weierstrass theorem and the Weierstrass criterion for the uniform convergence of series . He continued to lecture at the university even after his illness reduced him to a wheelchair. He died of pneumonia in 1897 . After his death all his writings and works were collected in seven volumes in Berlin in 1903.

[13]Heine Heinrich Eduard (1821 - 1881) was a German mathematician, known for his contributions to the theory of continuous functions. After following the lessons of Gauss, he had Dirichlet as a teacher. He is best known for the Heine-Borel and Heine-Cantor theorems. He is also responsible for the notion of uniform continuity. He also worked on Legendre polynomials, Lamé functions, Bessel functions, potential theory and partial differential equations.

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3. Analysis of the proof of the continuity of a sum of a series of continuous functions

To demonstrate the continuity of:

$$S(x) = \sum_{k=0}^{\infty}(x)$$

it should be noted that if $x$and$x_0$belong to the point-to-point convergence interval of the series, we have:

$$|S(x) - S(x_0)| =$$
$$= |s_n(x) - s_n(x_0) + R_n(x) - R_n(x_0)| < |s_n(x) - s_n(x_0) + R_n(x) + R_n(x_0)|$$

so to make the quantity $|S(x) - S(x_0)|$less than εneeds that over $|s_n(x) - s_n(x_0)|$also $|R_n(x)|$and $|R_n(x_0)|$can be increased by$\frac{\varepsilon}{3}$. This must happen for every$x$of the point wise convergence interval of the series and therefore from what has been said it follows that to affirm the continuity of the sum$S(x)$of the series it is necessary beyond the hypothesis of the continuity of the functions$a_k(x)$and of the convergence point of the series the supplementary one that for every$\forall\ \varepsilon > 0$ and for every $\forall\ x$of the point wise convergence interval of the series there exists an N belonging to the set of natural numbers such that for $n > N$it results $|R_n(x)| < \frac{\varepsilon}{3}$, which represents the hypothesis of uniform convergence of the series in the point-to-point convergence interval of the considered series. The hypothesis of uniform punctual convergence in addition to the punctual one is essential as can be deduced from the following example. Let's consider the series:

$$\sum_{k=0}^{\infty}(1-x)x^k \text{ in } I = \left|\frac{1}{3}\ ,1\right|$$

it is converging in $\left|\frac{1}{3}\ ,1\right|$because it is a geometric series that has $1 - x$as its first term and$x < 1$ as its reason. Therefore, it turns out:

$$S(x) = \frac{1-x}{1-x} = 1$$

moreover in $x = 1$we have$S(1) = 0$ since all its terms are equal to zero. Hence $S(x)$is discontinuous at $x = 1$. Observe that from the formula for the nth remainder we have:

$$|R_n(x)| = (1-x)x^n + (1-x)x^{n-1} + \cdots =$$
$$= (1-x)x^n(1 + x + x^2 + \cdots) = \frac{(1-x)x^n}{(1-x)} = x^n$$

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Therefore:

$$|R_n(x)| = x^n < \varepsilon \text{ per } n > \frac{\ln(\varepsilon)}{\ln(x)}$$

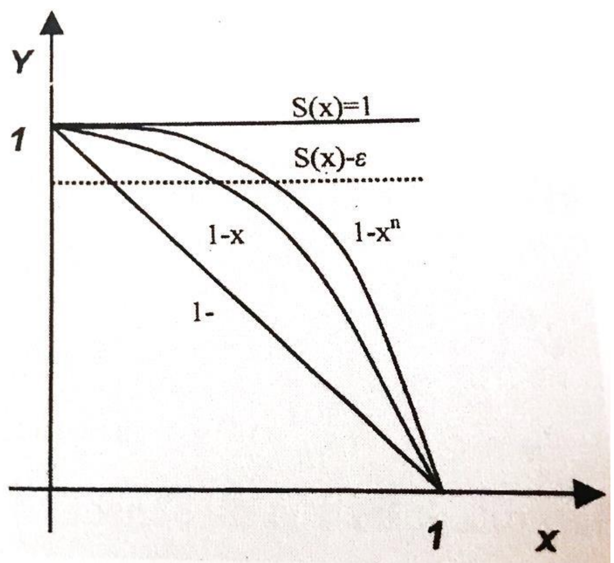
Since

$$\lim_{x \to 1^-} ln(x) = 0^+$$

We have:

$$\lim_{x \to 1^-} \frac{ln(\varepsilon)}{ln(x)} = +\infty$$

therefore, there is no Nsuch that for n> N it is $n > N$ sia $|R_n(x)| < \varepsilon$. The situation described above is highlighted in the following graph:



The plane strip bounded by $S(x)$ and $S(x) - \varepsilon$ does not entirely contain any$S_n(x)$and therefore the series does not converge uniformly.

4. The Cauchy proof

Let $\varepsilon > 0$be given and considered:

1) 1) Since every function is continuous, their partial sum$s_n(x)$is continuous:

$$\exists\ \delta\ \forall a |a| < \delta \Rightarrow |s_n(x+a) - s_n(x)| < \varepsilon$$

2) Since the series converges to x

$$\exists N \, \forall n > N |r_n(x)| < \varepsilon$$

3) Since the series converges to

$$x + a \quad \exists N \, \forall n > N |r_n(x+a)| < \varepsilon$$

as:

$$|s(x+a) - s(x)| =$$
$$= |s_n(x+a) + r_n(x+a) - s_n(x) - r_n(x)| \leq$$
$$\leq |s_n(x+a) - s_n(x)| + |r_n(x) + r_n(x+a)| \leq \varepsilon$$

In this way the function $s(x)$ is continuous.

Let us denote that in point 1) δ depends on ε, x and n, therefore 1) can be rewritten in this way: $\delta\ (\varepsilon, x, n)$; in step 2) N depends on ε and x therefore 2) can be rewritten in this way: $N(\varepsilon, x)$. In step 2) N depends on ε, x and a therefore 3) can be rewritten in this way: $N(\varepsilon, x + a)$. From the above we must know that:

$$M = max_t \, N(\varepsilon, t)$$

exists for every ε, and that M does not depend on x. So the additional hypothesis we need is the following:

$$\forall \varepsilon > 0 \quad \exists M \quad \forall n > M \Rightarrow |r_n(x)| < \varepsilon \quad \forall x$$

This additional hypothesis represents the condition of uniform convergence of the series.

Although with the incompleteness in the hypotheses, Cauchy's theorem has created new opportunities for study and research so it is not wrong to believe, in accordance with Lakatos' maxim, "that a correct concept can be generated by a non-rigorous proof" , and therefore that often a mistake can lead to significant progress in the mathematical sciences.

5. Conclusion

The contributions of the French mathematician have been really many not only in the purely mathematical field; in fact we find essential contributions also in the kinematics and mechanics of continuous systems. But the reputations of the great mathematicians are subject to the same vicissitudes that run through those of other great men and so after his death and even today, Cauchy was severely criticized for his production which was called "super production", often too hasty : his work includes 789 memoirs (some of which are very voluminous). A criticism of this kind is always unjustified when a man has produced a large quantity of first-rate works to which other minor ones are added, and is generally made by those who have produced relatively little, even if this "little" is not of superior quality as originality. Abel[14]defined Cauchy with these words: "*Cauchy is crazy, but he is the only one who knows how to do mathematics, and he is the only one who does pure mathematics today*". From this statement of this great and illustrious mathematician of the time, we can affirm with historical certainty that, Cauchy was one of the founding fathers of modern mathematics; and one of the most interesting features of the thought of the great French mathematician was the continuous search for rigor in all fields of mathematics. Cauchy's part in modern mathematics is undoubtedly a prominent part, as is universally admitted, albeit sometimes reluctantly.

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