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Investigation of Motion in a Central Field

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Abstract
The law of conservation of angular momentum lets us reduce problems about motion in a central field to problems with one degree of freedom. Thanks to this motion in a central field can be completely determined.

Keywords: A central field ; one degree of freedom; investigation of motion; Integration of the equation of motion

1. Introduction

We look at the motion of a point (of mass 1) in a central field on the plane:

$$\ddot{\mathbf{r}} = -\frac{\partial U}{\partial \mathbf{r}}, \; U = U(r)$$

It is natural to use polar coordinates r, φ.

By the law of conservation of angular momentum the quantity $M = \dot{\varphi}(t) r^2(t)$ is constant (independent of t).

Theorem . For the motion of a material point of unit mass in a central filed the distance from the center of the field varies in the same way as r varies in the one-dimensional problem with potential energy

$$V(r) = U(r) + \frac{M^2}{2r^2}$$

Proof. Differentiating the relation shown ($\dot{\mathbf{r}} = \dot{r} e_r + r\dot{\varphi} e_\varphi$) we find

$$\ddot{\mathbf{r}} = (\ddot{r} - r\dot{\varphi}^2) e_r + (2\dot{r}\dot{\varphi} + r\ddot{\varphi}) .$$

Since the field is central,

$$\frac{\partial U}{\partial \mathbf{r}} = \frac{\partial U}{\partial r} e_r .$$

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Therefore the equation of motion in polar coordinates takes the form

$$\ddot{r} - r\dot{\varphi} = -\frac{\partial U}{\partial r}, \; 2\dot{r}\dot{\varphi} + r\ddot{\varphi} = 0 .$$

But, by the law of conservation of angular momentum,

$$\dot{\varphi} = \frac{M}{r^2},$$

Where M is a constant independent of t, determined by the initial conditions.Therefore,

$$\ddot{r} = -\frac{\partial U}{\partial r} + r\frac{M^2}{r^4} \text{ or } \ddot{r} = -\frac{\partial V}{\partial r}, \text{ where } V = U + \frac{M^2}{2r^2} .$$

The quantity $V(r)$ is called the effective energy.

Remark The total energy in the derived one-dimensional problem

$$E_1 = \frac{\dot{r}^2}{2} + V(r) .$$

is the same as the total energy in the original problem

$$E = \frac{\dot{\mathbf{r}}^2}{2} + U(\mathbf{r}),$$

Since

$$\frac{\dot{\mathbf{r}}^2}{2} = \frac{\dot{r}^2}{2} + \frac{r^2\dot{\varphi}^2}{2} = \frac{\dot{r}^2}{2} + \frac{M^2}{2r^2} .$$

2. Integration of the equation of motion

The total energy in the derived one-dimensional problem is conserved.

Consequently, the dependence of r is defined by the quadrature

$$\dot{r} = \sqrt{2(E - V(r))}, \quad \int \mathrm{d}t = \int \frac{\mathrm{d}r}{\sqrt{2(E - V(r))}}$$

Since $\dot{\varphi} = \frac{M}{r^2}, \frac{\mathrm{d}\varphi}{\mathrm{d}r} = \frac{M/r^2 \,\mathrm{d}r}{\sqrt{2(E - V(r))}}$,and the equation of the orbit in polar coordinates is found by quadrature,

$$\varphi = \int \frac{M/r^2 \,\mathrm{d}r}{\sqrt{2(E - V(r))}}.$$

3. Investigation of the orbit

We fix the value of the angular momentum at M. The variation of r with time is easy to visualize, if one draws the graph of the effective potential energy $V(r)$ (Figure 1)

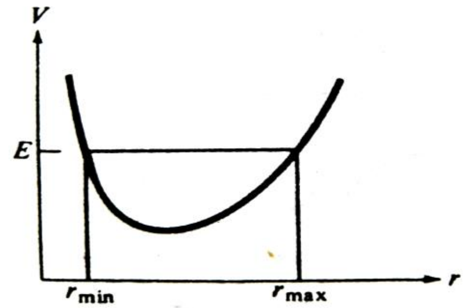

Figure 1 Graph of the effective potential energy

Let E be the value of the total energy. All orbits corresponding to the given E and M lie in the region $V(r) \le E$. On the boundary of this region, $V = E$,i.e., $\dot{r} = 0$.Therefore the velocity of the moving point, in general, is not equal to zero since $\dot{\varphi} \ne 0$ for $M \ne 0$.

The inequality $V(r) \le E$ gives one or several annular regions in the plane:

$$0 \le r_{\min} \le r \le r_{\max} \le \infty.$$

If $0 \le r_{\min} \le r \le r_{\max} < \infty$,then the motion is bounded and takes place inside the ring between the circles of radius $r_{\min}$ and $r_{\max}$.

The shape of an orbit is shown in Figure 2.

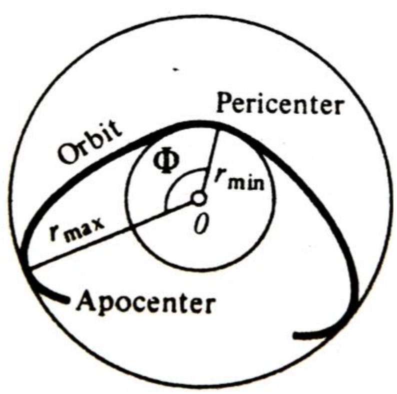

Figure 2 Qrbit of a point in a central field

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The angle φ varies monotonically while r oscillates periodically between $r_{\min}$ and $r_{\max}$ .The points where $r = r_{\min}$ are called pericentral, and where $r = r_{\max}$, apocentral (if the cebter is the earth-perigee and apogee; if it is the sun-perihelion and aphelion;if it is the moon-perilune and apolune).

Each of the rays leading from the center to the apocenter or to the pericenter is an axis of symmetry of the orbit.

In general, the orbit is not closed:the angle between the successive pericenters and apocenters is given by the integral

$$\Phi = \int_{r_{\min}}^{r_{\max}} \frac{M/r^2 \, \mathrm{d}r}{\sqrt{2(E - V(r))}} .$$

The angle between two successive pericenters is twice as big.

The orbit is closed if the angle Φ is commensurable with 2π ,i.e., if $\Phi = \frac{2\pi m}{n}$,where m and n are integers.

It can be shown that if the angle Φ is not commensurable with 2π ,then the orbit is everywhere dense in the annulus (Figure 3).

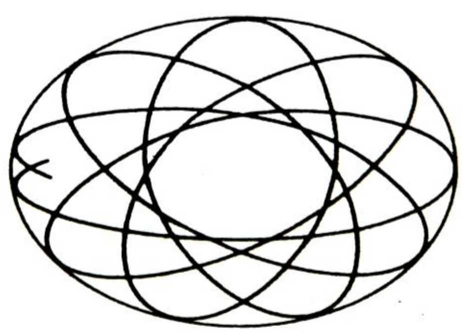

Figure 3 Orbit dense in an annulus

If $r_{\min} = r_{\max}$ .i.e., E is the value of V at a minimum point, then the annulus degenerates to a circle, which is also the orbit.

We now look at the case $r_{\max} = \infty$.If $\lim_{r\to\infty} U(r) = \lim_{r\to\infty} V(r) = U_{\infty}$,then it is possible for orbits to go off to infinity. If the initial energy E is larger than U_{∞} ,then the point goes to infinity with finite velocity $\dot{r}_{\infty} = \sqrt{2(E - U_{\infty})}$. We notice that if $U(r)$ approaches its limit slower than r^{-2} ,then the effective potential V will be attracting at infinity (here we assume that the potential U is attracting at infinity).

If , as $r \to 0, |U(r)|$ does not grow faster than $\frac{M^2}{2r^2}$,then $r_{\min} > 0$ and the orbit never approaches the center .If, however, $U(r) + \frac{M^2}{2r^2} \to -\infty$ the center of the field is possible even in finite (for example, in the field $U(r) = -\frac{1}{r^3}$).

4. Kepler's problem

This problem concerns motion in a central field with potential $U = -\frac{k}{r}$ and there $V(r) = -\frac{k}{r} + \frac{M^2}{2r^2}$ (Figure4),

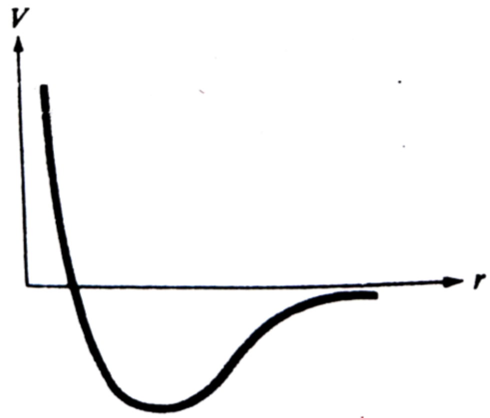

Figure 4 Effective potential of the Kepler problem

By the general formula

$$\varphi = \int \frac{M/r^2 \, \mathrm{d}r}{\sqrt{2(E - V(r))}}.$$

Integrating, we get

$$\varphi = \arccos \frac{\frac{M}{r} - \frac{k}{M}}{\sqrt{2E + \frac{k^2}{M^2}}}.$$

To this expression we should have added an arbitrary constant. We will assume it equal to zero; this is equivalent to the choice of an origin of reference for the angle φ at the pericenter. We introduce the following notation:

$\frac{M^2}{k}=p$, $\sqrt{1+\frac{2EM^2}{k^2}}=e$.

Now we get $\varphi=\arccos\frac{\frac{p}{r}-1}{e}$,i.e., $r=\frac{p}{1+e\cos\phi}$.

This is so-called focal equation of a conic section. The motion is bounded for $E<0$. Then $e<1$, i.e., the conic section is an ellipse. The number p is called the parameter of the ellipse, and e the eccentricity.

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