Solving the Contact Problem of Elasticity Theory in Modelling an Operating Tool for Tillage

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Abstract
Viewing soil as a medium with linear elastoviscous deformation, a mathematical model is proposed, which makes it possible to develop a toothed tool with such a shape of the working surface that satisfies the agrotechnical, technological and economic indicators when soil tilling.
The operating tool is made in the form of a block of teeth. The valley and the protrusion of the tooth in the horizontal plane are made along a logarithmic spiral, and the protrusions are made in the form of a fourth-degree parabola.
The toothed tool equation is presented as a combination of rotation, displacement and compression matrices.

Keywords: soil, valleys and protrusions of teeth, logarithmic spiral, fourth-order parabola, linear elastoviscous deformation

The fertility of black earth soils, as the main economic resource of Ukraine, depends on the quality of their processing. It is known that loosening is the key criterion of soil tillage quality. The level of soil loosening depends on the geometry of the operating tool and the kinematics of its movement in the work process. Studies have shown that the operating tool's geometry and its movement's kinematics affect the energy output of soil treatment, as well as the stress-strain state of the tillable soil and the tool during its operation. To meet the above requirements for soil treatment, research on the interaction of the operating tool with the soil was carried out. When doing so, the soil can be viewed as a medium with linear elastoviscous deformation. In the event of contact of linear-elastoviscous bodies, the issue of creating a mathematical model of the

Vol. 10, No. 02; 2025

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working body is reduced to solving problems by methods of elasticity theory based on the "correspondence principle".

As a test object, it is recommended to initially take an operating tool for soil treatment, made in the form of a block of teeth.

The interaction of the toothed operating tool with the soil can be represented as the interaction of two bodies at the contact site described by the integral equation of the contact problem (1)

$$\int_{-a}^{a} P(t)\ln\frac{1}{|t-x|}\,dt = f(x), \quad (1)$$

where $f(x)$ is function specified within interval $(-a,a)$ and depending on the shape of the operating tool in the horizontal plane and deformative constants of the operating tool and soil.

This function can be determined from the equation

$$f(x) = \frac{C - f_1(x) - f_2(x)}{\nu_1 + \nu_2}, \quad (2)$$

where $f_1(x), f_2(x)$ are functions describing the configuration of the operating tool and soil in the horizontal plane; C is some constant; ν_1, ν_2 are deformable constants of soil and the operating tool.

To solve the problem of the interaction of a toothed operating tool with the soil, the following restrictions should be taken:

1. The soil is schematically represented as a half-plane, which leads to the following features of the geometry of the soil surface

$$f_1(x) = f_1'(x) = f_1''(x) = x. \quad (3)$$

2. The deformable constant of the operating tool ν_2 can be neglected $\nu_2 \to 0$, since the rigidity of the operating tool is several times higher in comparison with the soil. So, the modulus of elasticity of soils does not exceed $50-80 M\Pi a$. Whereas for metals the modulus of elasticity is $2 \cdot 10^5 M\Pi a$.

The deformable constants of two media can be written as

$$\nu_1 = \frac{2}{\pi \cdot E_{\partial 1}}\left(1-\mu_{b_1}\right), \quad \nu_2 = \frac{2}{\pi \cdot E_{\partial 2}}\left(1-\mu_{b_2}\right), \quad (4)$$

where $E_{\partial 1}$ is the deformation modulus for the linearly deformable first medium (material of the working body – steel); $E_{\partial 2}$ is the deformation modulus for a linearly deformable second medium

Vol. 10, No. 02; 2025

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(soil); μ_{b1} is the lateral expansion coefficient of steel; μ_{b2} is the lateral soil expansion coefficient.

It is known from the elasticity theory that there is a relation between the shear modulus and the modulus of elasticity through Poisson's ratio

$$E = 2(1-\mu_b)G, \quad (5)$$

therefore

$$\mu_b = \frac{E}{2G} - 1.$$

Using the "correspondence principle" for a linearly deformable medium, it is possible to write

$$\mu_{b2} = \frac{E_\partial}{2G_\partial}, \quad (6)$$

where $E_\partial = E$ is the deformation modulus equal to the elastic modulus.

Since $\nu_1 \to 0$, then ν_2 when substituting (6) in (4) will be equal to

$$\nu_2 = \frac{1-\mu_{b2}}{\pi \cdot G_\partial}. \quad (7)$$

The pressure distribution law for each contact area of the toothed operating tool depends on the shape of the area. When a toothed operating tool with wedge-shaped teeth acts on the soil (the equation of the wedge configuration $y_2 = f_2(x) = Ax$), the pressure distribution is expressed by the equation

$$P(x) = -\frac{P}{\pi a} \ln \frac{a - \sqrt{a^2 - x^2}}{|x|}, \quad (8)$$

where $P(x)$ is an unknown function within the interval $(-a, a)$, satisfying equation (1); a is contact half-width; P is the force applied to the operating tool.

The maximum pressure, equal to infinity, develops at the tip of the wedge at (Figure 1.c). According to the literature data [1, 4], the process of cracking in the soil depends on the nature of the pressure distribution in the contact area. Primary cracks occur at points of maximum pressure. In this case, the primary crack occurs at the tip of the wedge.

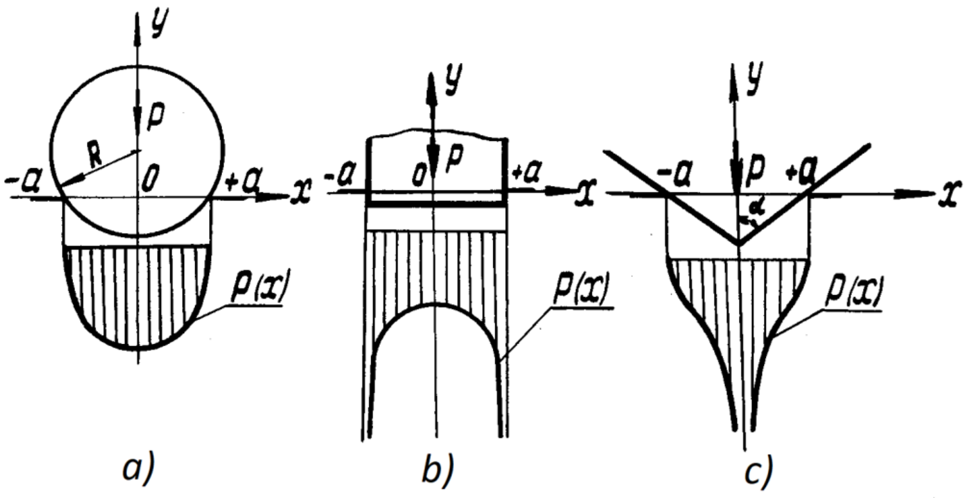


Figure 1 – Distribution of pressure of teeth of various shapes: a) round; b) flat; c) wedge-shaped.

In case of rectangular contact areas, a pressure concentration is created, causing a stress-strain state in a limited volume of soil. Thus, the pressure distribution is expressed by the equation

$$P(x) = \frac{P}{\pi\left(\sqrt{a^2 - x^2}\right)}. \quad (9)$$

Analysing equation (9), it is easy to see that the pressure at the edges of the rectangular contact areas of the toothed operating tool with $x = \pm a$ is maximum and tends to infinity (Figure 1.b). Cracks form and develop at the edges of the contact area.

The round shape of the protrusions of the toothed operating tool with radius R will have the following pressure distribution

$$P(x) = \frac{2P}{\pi \cdot a^2}\sqrt{a^2 - x^2}. \quad (10)$$

The maximum pressure develops at the point of initial contact, i.e. at $x = 0$, and equals

$$P(0) = \frac{1}{\pi}\sqrt{\frac{2\pi}{R \cdot v_2}}. \quad (11)$$

The circular contact section of the toothed operating tool, in comparison with the rectangular and wedge-shaped, has a more uniform pressure distribution (Figure 1.a). This pressure distribution causes the formation of several cracks with a more developed central one.

When various forms of teeth interact with the soil, due to the uneven distribution of contact pressures on the surface, the cracking process does not occur over the entire contacting surface, affecting the quality of soil treatment. Teeth made in the form of a wedge, a rectangle, or a ball wear out during operation. Areas with maximum contact pressure are subject to intense wear.

The wear process is not even. As a result, it is necessary to have such a shape of teeth that, when interacting with the soil, uniform pressures appear on the contact surface. This will entail evenly distributed cracking in the soil. This improves its loosening and reduces tool wear.

Suppose that the functions $f_1(x)$ and $f_2(x)$, which determine the configuration of the operating tool and the soil in the horizontal plane, have continuous first and second derivatives. In the neighbourhood of point $x=0$ (1)

$$f_1'(x) = f_2''(x) = 0. \quad (12)$$

Regarding the forces compressing the bodies, we will assume that their resultant perpendicular axes ox, are directed to the point of initial contact of the compressible bodies, i.e., to coordinate origin. Since the initial clearance between the compressible bodies $f_1(x)+f_2(x)$ is assumed to be symmetrical about the axis oy, the pressure on the surfaces of the compressed bodies will also be symmetrical about the axis oy.

Consider the case when the relation determines the sum of the second derivatives

$$f_1''(0) + f_2''(0) = 0. \quad (13)$$

Suppose that not only the second derivative of the sum $f_1(x)+f_2(x)$, but all subsequent derivatives up to $2n-1$ inclusive, vanish at $x=0$. Meanwhile, derivative $f_1^{(2n)}(x)+f_2^{(2n)}(x)$ is nonzero at $x=0$, being continuous in this point. In this case, considering the smallness of the contact area with $-a \le x \le a$, we can approximately assume

$$f_1(x) + f_2(x) = \frac{1}{(2n)!}\left[f_1^{2n}(0) + f_2^{2n}(0)\right] \cdot x^{2n}. \quad (14)$$

Substituting (14) in (2) we find

$$f(x) = \alpha - A_n x^{2n}, \quad (15)$$

where

$$A_n = \frac{f_1^{(2n)}(0) + f_2^{2n}(0)}{(2n)!(\nu_1 + \nu_2)}. \quad (16)$$

The solution to the main integral equation (1) of the contact problem for the case when the right-hand side is represented in the form (15) is

$$P(x) = \frac{P}{\pi \cdot a^2}\sqrt{a^2 - x^2} \cdot \left[\frac{2n}{2n-1} + \frac{2n(2n-2)}{(2n-1)(2n-3)} \cdot \frac{x^2}{a^2} + \frac{2n(2n-2)\ldots \cdot 2}{(2n-1)(2n-3)\ldots \cdot 3 \cdot 2 \cdot 1} \cdot \frac{x^{2n-2}}{a^{2n-2}}\right]. \quad (17)$$

We assume that a constant force P affects the operating tool, and the half-width of the contact area a is known. Solving equation (17), we obtain the pressure distribution under the section of the toothed working body, the shape of which is a parabola of even degrees when changing n

from 1 to 5. We obtain a uniform pressure distribution at $P(x) = const$. Using this ideal case, we will measure the actual pressure distributions by comparing the deviations of the obtained distributions with different n, in terms of the standard deviation from the mean (Figure 2). At the minimum value σ we obtain the optimal value n and the corresponding distribution $P(x)$ (Figure 3).

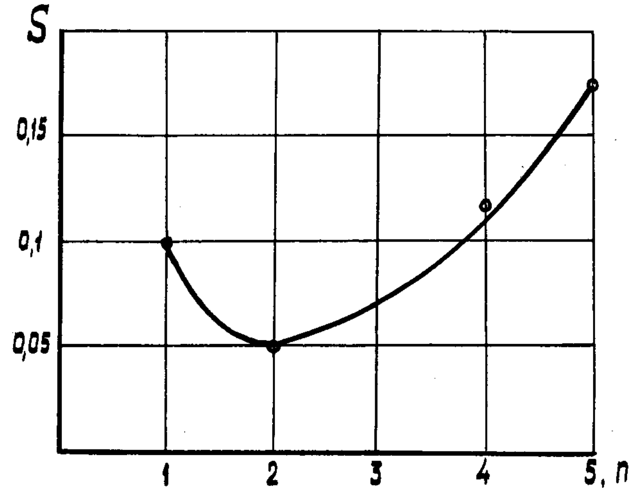
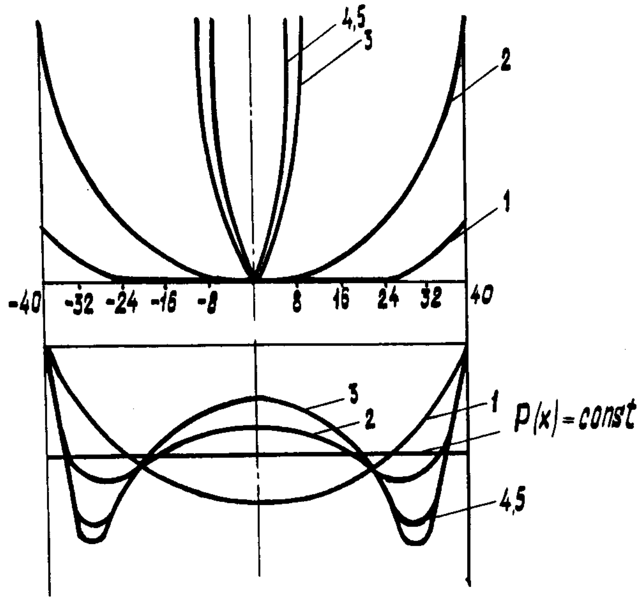


Figure 2 – Deviations from the mean depending on the exponent.



Picture 3 – Distribution of tooth pressures depending on the exponent.

Vol. 10, No. 02; 2025

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The results of theoretical studies show that to obtain a more uniform distribution of contact pressures while the operating tool acts on the soil, the shape of the protrusions should be a fourth-degree parabola with the vertex directed towards the movement of the operating tool. In this case, the formation of cracks will occur more evenly over the entire contacting surface of the protrusions. The pressure on the contact area is as follows

$$P(x) = \frac{4P}{3\pi \cdot a^4}\left[\left(a^2 + 2x^2\right)\sqrt{a^2 - x^2}\right] \quad . \qquad (18)$$

The task of designing a toothed operating tool for tillage machines is reduced to the choice of a mathematical model that would meet all the demands made to the operating tool in technological and economic terms, considering the physical and mechanical properties of the soil.

The operating tool is made in the form of a block of teeth (Fig. 1), constituting valleys in the horizontal plane made along a logarithmic spiral and protrusions profiled along the fourth-degree parabola.

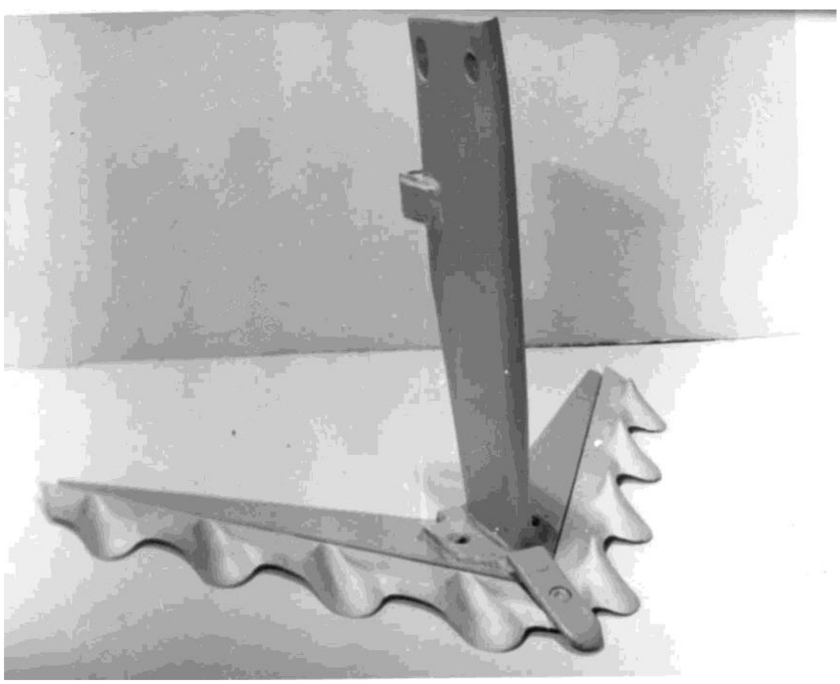

Figure 4 – General view of the toothed operating tool for tillage.

In horizontally projecting planes, the section of the tooth is a family of logarithmic spirals

$$r_{\psi_i} = r_0 e^{\psi \, tg\varphi}, \qquad (19)$$

where r_{ψ_i} is current radius vector; r_0 is initial radius vector; $tg\phi$ is internal friction coefficient; ψ is the current angle of the radius vector of the spiral.

The equation of the surface of the toothed operating tool for tillage can generally be represented using the matrix

$$A_{ij} = \left(\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ \hline 0 & 0 & 0 & 1 \end{array}\right), \qquad (20)$$

where $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ is a third-order matrix describing rotation in three-dimensional space; $(a_{14}, a_{24,} a_{34})^T$ are translational components.

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